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Holographic QCD and pion mass

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Abstract: To realize massive pions, we study variations of the holographic model of massless QCD using the D4/D8/$\overline{\text{D8}}$ brane configuration proposed by Sakai and Sugimoto. We propose deformations which break the chiral symmetry explicitly and compute the mass of the pions and vector mesons. The observed value of the pion mass can be obtained. We also argue a chiral perturbation corresponding to our deformation.

Keywords: AdS-CFT Correspondence, Brane Dynamics in Gauge Theories, QCD.

Contents

1. Introduction

Holographic approach in analyses of QCD as an application of the AdS/CFT correspondence [1] is not only a way to study strong coupling dynamics of QCD in a perturbative way, but also giving a hope to find a more comprehensive and deeper understanding of hadron physics and holography [2] itself. The developments in this research subject are still in the middle of the way to go beyond the large N approximation and to include dynamical quarks and various interactions. A lot remain to be studied and to be revealed important and intriguing.

Among many gravity models holographically dual to QCD, proposed so far (see for example [3, 4]), the Sakai-Sugimoto model [5] is one of the most successful models at present. An important feature of the model is that, in terms of D-brane geometries, it explains the spontaneous chiral symmetry breaking occurring at strong coupling in the real QCD. This is a typical example where D-branes in string theory provide a new interpretation of known physics — any new interpretation may help to create new techniques for analyzing physical systems. Sakai and Sugimoto predicted various important phenomenological parameters associated with hadron physics, such as vector/scalar meson spectra, interactions among them, chiral Lagrangian with calculable coefficients, skyrmions, and so on. The comparison to the measured observable values is quite successful, the values turned out to be within $20-30\%$ error [5].

The Sakai-Sugimoto model deals with massless QCD, thus the chiral symmetry is unbroken at weak coupling. The spontaneous chiral symmetry breaking is realized in terms of D-branes, and massless pions appear as fluctuations on the D-branes. In the real world, the quarks are massive and the chiral symmetry is explicitly broken, thus the pions are pseudo Nambu-Goldstone bosons. To study the contribution of quark masses to hadron physics, we have to introduce quark masses to the holographic QCD model. Since the holographic QCD model describes hadron physics, we aim in this paper to provide a D-brane construction to introduce the pion mass in the Sakai-Sugimoto model, and to see how the hadron dynamics described in the massless model is modified in the presence of the pion mass. A smooth limit to the Sakai-Sugimoto model shows the existence of a corresponding chiral perturbation giving the pion mass.

In the Sakai-Sugimoto model, left-handed (or right-handed) quarks live on the intersection point of "gauge" N_c D4-branes and "flavor" N_f D8-branes (or $\overline{\text{D8}}$-branes). The quark mass term mixes the left and the right, thus we need to bend the D8-branes and the $\overline{\text{D8}}$-branes and connect them *even at the weak coupling regime*, i.e. even as a D-brane configuration in the flat spacetime background. This can be achieved by introducing a throat configuration of D8-$\overline{\text{D8}}$ branes [6] and placing D4-branes inside the throat, which will be studied in section 2. Our result is that the throat surface is located outside the near-horizon region of the D4-branes and furthermore the pions are somehow still massless. Then we consider a particular limit of this brane configuration, to make the D8-brane throat be flat and parallel to the N_c D4-branes, and put it inside the near horizon region. In this limit, the pion becomes massive as expected, however it is too heavy (with no massless pion limit), and so this brane configuration is not phenomenologically viable. In section 3, we consider a different approach, which is introduction of a bound D4-brane charge on the D8- and the $\overline{\text{D8}}$-branes of the Sakai-Sugimoto model. It is given by a Yang-Mills instanton on angular S^4 of the probe D8-brane worldvolume in the near-horizon background geometry.[1] This breaks the chiral symmetry explicitly, and with this, the value of the pion mass is successfully tuned to be the realistic one.[2] We work out fluctuation analysis of the probe D8-brane in the background geometry, following the computations in [5]. We find that vector meson spectrum obtained in [5] is not significantly affected by the introduction of the pion mass. We discuss a possible chiral perturbation corresponding to this introduction of the D4-brane charge.

2. Toward a holographic dual of QCD with massive quarks: some attempts

In the Sakai-Sugimoto model, the pure Yang-Mills part of the QCD is realized as a four-

[1] In the paper [7], Yang-Mills instantons are introduced as candidates for the baryons. In their context, the instanton is particle-like in four-dimensional spacetime in which the QCD is realized. On the other hand, in our model, the instanton will not be localized in any direction of this four-dimensional spacetime.

[2] The contribution of the instanton do not give the η' mass. For the origin of the η' mass, see the first reference of [5].

dimensional effective theory on N_c D4-branes compactifying one spatial direction by an S^1 with imposing the anti-periodic boundary condition for fermionic fields. They introduce N_f D8-branes and N_f $\overline{\text{D8}}$-branes which are located at distinct points in the S^1 direction, and an open string stretching from the D4-branes to the D8- or $\overline{\text{D8}}$-branes provides a chiral or anti-chiral fermion in four dimensions. The chiral $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ symmetry is realized as a direct product of gauge symmetries on the N_f D8- and N_f $\overline{\text{D8}}$-branes. In the gravity dual description, where the D8- and $\overline{\text{D8}}$-branes can be treated as probes in the $N_f \ll N_c$ limit, the D8- and $\overline{\text{D8}}$-branes have to connect with each other smoothly in the near horizon geometry of the corresponding D4-brane. This is interpreted as the spontaneous chiral symmetry breaking in QCD, and they show the existence of Nambu-Goldstone bosons associated with the spontaneous chiral symmetry breaking, i.e. pions. However since the D4-branes intersect with the D8- and $\overline{\text{D8}}$-branes, the quark masses are zero and then the pions are exactly massless.

We expect that a quark becomes massive if we can deform the D8- and $\overline{\text{D8}}$- brane configuration in such a way that they do not intersect with the D4-branes in the flat spacetime. Although it is an unstable configuration, it is known that there is such a static configuration in which the parallel D8- and $\overline{\text{D8}}$-branes are connected by a throat with the size almost equal to the asymptotic distance between the D8- and $\overline{\text{D8}}$- branes [6], see figure 1. We then place D4-branes inside the throat. Because the D8- and $\overline{\text{D8}}$-branes no longer intersect with the D4-branes, and also because the chiral symmetry is explicitly broken due to the fact that the D8- and $\overline{\text{D8}}$-branes are already connected, the masses of the quarks are expected to be non-zero and proportional to the size of the throat radius, thus the pions become pseudo Nambu-Goldstone bosons.[3]

The holographic dual gravity description for this D-brane throat should be a new D8-brane probe configuration in the same D4-brane geometry. In the flat spacetime, for a fixed distance between the D8- and the $\overline{\text{D8}}$-branes, there are two static configurations as we have already mentioned: the flat D8- and $\overline{\text{D8}}$-branes, and the throat configuration. The former corresponds to the original Sakai-Sugimoto model, while the latter is what we are interested in. Thus, we expect that when the D4-branes are replaced by their curved background geometry, we would have two probe configurations, one for the former and the other for the latter. However, the analysis in [5] shows that, at least in the near-horizon region, there is a unique probe configuration with the asymptotic distance between the D8- and the $\overline{\text{D8}}$-branes fixed to be the anti-podal points in the S^1.[4]

In order to find the missing solution, we study the probe D8-brane configuration in the full D4-brane geometry without taking the low energy limit, i.e. the near horizon limit. Now the boundary condition for the D8-brane solution is imposed at the asymptotically

[3]We notice that a tachyonic mode appears in the scalar field on D8-branes, although this mode does not correspond to the pion.

[4]In the paper [8], a one-parameter family of the D8-brane probe configuration in the near-horizon geometry of the D4-branes is obtained. The parameter is the asymptotic distance between the D8- and the $\overline{\text{D8}}$-branes in the S^1. The D8- and $\overline{\text{D8}}$-branes still intersect with the D4-branes in the weak coupling picture (i.e. in the flat spacetime), thus they have massless Nambu-Goldstone bosons in the strong coupling picture.

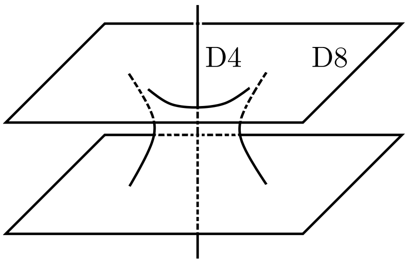

Figure 1: The D4-branes are in the throat of the D8 $\overline{\mathrm{D8}}$.

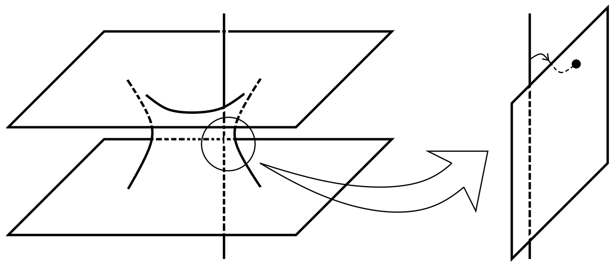

Figure 2: The D4-brane location is shifted, and the large throat limit is taken.

flat region. Then we find two probe D8-brane solutions for the same anti-podal boundary condition: one extends into the near horizon geometry and the other does not. The former solution corresponds to the D8-brane considered in Sakai-Sugimoto model, and we identify the latter as what we are looking for. Although the fields on D8-branes do not decouple with bulk gravity modes and string massive modes (since we do not take the near horizon limit), we may expect those modes do not break the chiral symmetry and the pions are still massless if the quarks are massless. Conversely if there are no massless pions, we may understand that the chiral symmetry is explicitly broken due to the non-zero quark mass terms. We studied the fluctuation on this D8-probe configuration to check if pions become massive. On the contrary to our expectation from the D-brane picture in the flat space, we find the pions are still massless (see more detail in appendix A).

One possible explanation for this situation is that without taking the near horizon limit, above picture just fails to capture the strong dynamics of the dual QCD. However it may also be understood in the following way. The amount of difference in energy from the D-brane configuration in [5] is finite, which might imply that we are looking at an excited state in the same theory (where quarks are massless) and not a vacuum of a different theory. The operator corresponding to the scalar field on the D8-brane, which determines the probe D8-brane configuration, is not a quark bi-linear (the quark mass term), but a four fermi term [9] which can connect left- and right-handed quarks without breaking the chiral symmetry. Nevertheless, this is still counter-intuitive, because the open string stretching between the D8- and the D4-branes provides a quark field and the energy of this open string is proportional to the distance between these two sets of D-branes. In order to see this property more appropriately, we come to consider a limit of the above D-brane configuration: taking the large radius limit of the throat, and placing the D4-branes inside the throat not at the center, but at some fixed distance from the throat surface. We "magnify" the throat region of the D8-branes, while keeping the distance to the D4-branes (see figure 2). After taking this limit the D8-branes are placed parallel to the D4-branes. Then it is clear that the quarks are massive, since both the D4- and the D8-branes are flat and separated by a non-zero distance. Although there is no chiral symmetry because of the non-zero masses for the quarks, we expect there are pseudo Nambu-Goldstone bosons if the distance between the D4- and the D8-branes is much smaller than the QCD scale.

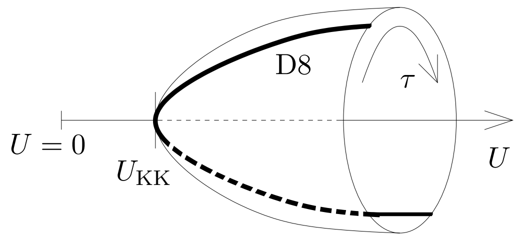

Figure 3: The D8-brane configuration localized at the anti-podal points on the S^1.

However this limit causes a problem. The five dimensional theory before compactification is a supersymmetric Yang-Mills theory with N_f hypermultiplets. Therefore with the S^1 compactification with the supersymmetry breaking boundary condition, the quark mass terms are at least radiatively generated[5] due to the lack of the chiral symmetry, even when the bare masses for the quark fields are zero. Then we do not expect the pseudo Nambu-Goldstone bosons. Despite of this, we might still expect that the mass of the pions is smaller than the masses of other mesons and that the mass spectrum is close to the meson spectrum in the actual QCD. With this in mind, we numerically computed the pion and the ρ meson masses, and found that the ratio is around 0.8. This mass is clearly too large as the real pions in QCD. We give more detailed calculations in appendix B.

3. D4-brane charge and pion mass

In the previous section, aiming to separate the D8-branes from the D4-branes, we studied deformations of the probe D8-brane configuration. In this section we study a deformation expressed by a non-trivial background of gauge fields on the flavor D8-branes. In particular we introduce a D4-brane charge on the D8-branes. One motivation for introducing the D4-brane charge is that the conservation law of it requires that the D8- and the $\overline{\rm D8}$-branes are connected to each other, even in the flat spacetime background (i.e. in the weak coupling regime of the QCD).[6] From the view point of the effective field theory on the D8-branes, this D4-brane is described by an instanton in an angular S^4 of the worldvolume of the D8-branes and thus breaks the chiral symmetry which is realized as a gauge symmetry on the D8-branes. It may also be important that this deformation by introducing the D4-brane charge on the S^4 extends infinitely along the direction orthogonal to the color D4-branes since it implies the explicit chiral symmetry breaking. From these perspectives, we expect that the pions become massive and, as we will see, we indeed realize a phenomenologically acceptable value of the pion mass.

The hadron physics in QCD is holographically captured by the probe D8-branes in the D4-brane geometry in the Sakai-Sugimoto model. In the following, we take the small α' limit keeping the magnitude of the field strength F so that $\alpha' F$ becomes small. Since the

[5]This system was first studied by the paper [10]. The quark field can be periodic along the S^1 (the scalar partner is then anti-periodic) in the field theoretical view point.

[6]This "thin throat" configuration of the D8-$\overline{\rm D8}$ is the one considered in [6] where a fundamental string charge is introduced instead of our D4-brane charge.

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gauge field appears in the DBI action and also in the Wess-Zumino term as sub-leading order terms in α', the back-reaction from the instanton to the shape of the D8-brane is sub-leading in the above limit. Hence we will consider the same D8-brane configuration as [5] as the leading order solution in the α' expansion. With this D8-brane configuration, the action of the gauge field is provided by the Yang-Mills action on the given D8-brane background.[7] In particular we concentrate on the case with $N_f = 2$. In subsection 3.1 we start with a brief review of a description of the probe D8-branes used in [5] and then introduce the instanton solution. After that, in subsection 3.2, we analyze spectra of fluctuations of the gauge fields around this instanton background, which corresponds to spectra of mesons appearing in the strong coupling regime of QCD. We show that the instanton background successfully gives non-zero masses to the pions. Finally in subsection 3.3 we give an interpretation of the introduction of the instanton as a chiral perturbation in QCD.

3.1 Instanton on probe D8-brane

We are interested in probe D8-branes extended into the near horizon region of the non-extremal D4-brane background [11, 4] given by

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{3/2}\left(dx_4^2 + f(U)d\tau^2\right) + \left(\frac{R}{U}\right)^{3/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \\ e^{\phi} &= \left(\frac{U}{R}\right)^{3/4}, \quad F_4 = \frac{2\pi N_c}{V_4}\epsilon_4, \quad f(U) = 1 - \frac{U_{\mathrm{KK}}^3}{U^3}. \end{aligned} \tag{3.1}$$

Here $dx_4^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ is the line element of the four-dimensional Minkowski spacetime and $d\Omega_4^2 = h_{ij}d\theta^i d\theta^j$ is that of a unit S^4. The four form ϵ_4 and the value V_4 are the volume form and the volume of a unit S^4. The constant g_s is the string coupling and the parameter R is related to the number N_c of the color D4-branes. The range of U is limited by $U \geq U_{\mathrm{KK}}$ and the τ-direction is compactified on an S^1 with the periodicity $\delta\tau = 4\pi R^{3/2}/3U_{\mathrm{KK}}^{1/2}$. For fermions, we impose the anti-periodic boundary condition along this S^1. With the periodicity $\delta\tau$, $\tau - U$ plane is locally flat near $U = U_{\mathrm{KK}}$ without any conical singularity. The coordinates x^μ and τ are the ones along the D4-brane world volume.

The probe D8-brane worldvolume is on a plane defined by a constant τ. This is a solution since the metric does not depend on τ and the D8-branes are placed at the anti-podal points on the S^1, see figure 3. With use of a new coordinate z defined by $U^3 = U_z^3 \equiv U_{\mathrm{KK}}^3 + U_{\mathrm{KK}}z^2$, the induced metric on the D8-branes can be written down as

$$ds_{D8}^2 = g_{MN}d\sigma^M d\sigma^N = \left(\frac{U_z}{R}\right)^{3/2} dx_4^2 + \frac{4}{9}\left(\frac{R}{U_z}\right)^{3/2}\frac{U_{\mathrm{KK}}}{U_z}dz^2 + \left(\frac{R}{U_z}\right)^{3/2} U_z^2 d\Omega_4^2. \tag{3.2}$$

Here indices M and N run from 0 to 8.

As explained at the beginning of this section, we study the Yang-Mills action on the above D8-brane solution in the small α' limit:

$$S_{D8} = T_{D8}(2\pi\alpha')^2\int d^9\sigma e^{-\phi}\sqrt{-\det g}\,\frac{1}{2}\mathrm{Tr}F_{MN}F^{MN}. \tag{3.3}$$

[7] Note that on this given D8-brane background, the Wess-Zumino term becomes total derivative and do not affect neither the instanton solution nor the spectrum analysis bellow.

Here T_{D8} is the tension of a D8-brane and the field strength F_{MN} is defined by $F_{MN} \equiv \partial_M A_N - \partial_N A_M + [A_M, A_N]$. (We use the convention in which the gauge fields $A_M = iA_M^a T^a$ are anti-Hermitian matrices and the generators are normalized as $\mathrm{Tr}T^aT^b = (1/2)\delta^{ab}$.) We introduce an instanton solution of this action and analyze its effects on the meson spectrum. After the Kaluza-Klein (KK) reduction to four dimensions, this action describes pions and vector mesons which are realized as KK modes of the gauge fields. The instanton is introduced on the S^4 and then the worldvolume of the induced D4-brane extends in the non compact four dimensions x^μ and also in the z-direction. The equations of motion for the gauge fields can be solved by the ansatz

$$A_\mu = 0, \quad A_z = 0, \quad A_i = A_i(\theta^j), \tag{3.4}$$

with the self-dual conditions $F_{ij} = *F_{ij}$. Here $*$ is the Hodge dual on a unit S^4. A solution of this self-dual equation can be obtained from the SU(2) instanton solution on a flat $\mathbf{R}^4$ [12]:

$$A_a^{\mathrm{inst}}(X) = i\frac{\epsilon_{abc}X^c\sigma_b - X^4\sigma_a}{\mu^2+\rho^2}, \quad (a=1,2,3), \quad A_4^{\mathrm{inst}}(X) = i\frac{X^a\sigma_a}{\mu^2+\rho^2}, \tag{3.5}$$

by the stereographic projection:

$$\begin{aligned} &X^1 \equiv \cot\left(\theta^1/2\right)\cos(\theta^2), && X^2 \equiv \cot\left(\theta^1/2\right)\sin(\theta^2)\cos(\theta^3), \\ &X^3 \equiv \cot\left(\theta^1/2\right)\sin(\theta^2)\sin(\theta^3)\cos(\theta^4), && X^4 \equiv \cot\left(\theta^1/2\right)\sin(\theta^2)\sin(\theta^3)\sin(\theta^4). \end{aligned} \tag{3.6}$$

From the Jacobian of this transformation we have $16d^4X = (\rho^2+1)^4 d\Omega_4$. The coordinate systems $\{X^a, X^4\}$ and $\{\theta^i\}$ are the ones for a flat $\mathbf{R}^4$ and a unit S^4 respectively, and the coordinate $\rho \equiv |\vec{X}|$ is the radial coordinate in the $\mathbf{R}^4$. The matrices $\{\sigma_a\}$ are the Pauli matrices. The parameter μ controls the size of the instanton. The coordinates X^a, X^4 and the parameter μ are all dimensionless in our notation. When we map the instanton solution (3.5) with $\mu = 1$ onto an S^4, we have a homogeneous instanton [13] which satisfies $\mathrm{Tr}F_{ij}F^{ij}$ = constant. In this case, the combined solution (3.2) and (3.5) is not only the solution of the above Yang-Mills theory, but also the solution of the original DBI action. For $\mu \neq 1$, the instanton number density $\mathrm{Tr}F_{ij}F^{ij}$ becomes inhomogeneous and dependent on θ^1, or equivalently on ρ.

3.2 Meson spectrum

We now study the fluctuations of the gauge fields around the instanton background and perform the KK reduction to four dimensions. In [5], only the lowest modes with respect to the S^4-coordinate dependence are considered, and mixing terms between A_μ (or A_z) with A_i vanish. In our case with an instanton, the mixing terms do not vanish. However in this subsection, we assume that the mixing between fluctuations of A_μ or A_z with that of A_i is small, and concentrate on the effect of the instanton on the A_μ-A_z system. We discuss the mixing terms with A_i in appendix C.[8]

[8] In appendix C, we show that the lowest S^4 KK modes of A_i do not mix with those of A_μ nor A_z. Although we have not evaluated the mixing terms including higher S^4 KK modes we expect that these terms are suppressed because of the small overlap between the wavefunctions.

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Plugging the induced metric (3.2) into the action (3.3) we obtain the four-dimensional Lagrangian:

$$S_{D8} = \tilde{T}(2\pi\alpha')^2 \int d^4x \mathcal{L}, \tag{3.7}$$

$$\begin{aligned}\mathcal{L} = \int dz \frac{d\Omega_4}{V_4} 2\mathrm{Tr}\Big\{ &\frac{R^3}{4U_z}\eta^{\mu\nu}\eta^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} + \frac{9}{8}\frac{U_z^3}{U_{\mathrm{KK}}}\eta^{\mu\nu}F_{\mu z}F_{\nu z} \\ &+ \frac{1}{2}\eta^{\mu\nu}h^{ij}D_iA_\mu D_jA_\nu + \frac{9}{8}\frac{U_z^4}{R^3U_{\mathrm{KK}}}h^{ij}D_iA_zD_jA_z + (\text{terms with } A_i)\Big\}.\end{aligned} \tag{3.8}$$

Here $\tilde{T} = 2R^{3/2}U_{KK}^{1/2}T_{D8}V_4/3$, and D_iA_μ and D_iA_z are defined by $D_iA_M \equiv \partial_iA_M + \left[A_i^{\mathrm{inst}}, A_M\right]$ (where $M = \mu$ or z). We first perform the integration over the S^4. Recalling that the instanton number density depends only on $\rho = |\vec{X}|$, we assume that the wave functions of the lowest S^4 KK modes of A_μ and A_z depend only on ρ accordingly. Under this assumption, we can write these lowest modes as follows:

$$A_\mu(x,z,X) = \tilde{A}_\mu(x,z)\zeta(\rho), \quad A_z(x,z,X) = \tilde{A}_z(x,z)\zeta(\rho). \tag{3.9}$$

We define $\zeta(\rho)$ by the eigenvalue equation (with the lowest eigenvalue ϵ^2)

$$-\partial_\rho\left(\rho^3L\partial_\rho\zeta\right) + 8L\frac{\rho^5}{(\mu^2+\rho^2)^2}\zeta = \rho^3L^2\epsilon^2\zeta, \quad \left(L \equiv 4(\rho^2+1)^{-2}\right), \tag{3.10}$$

and the normalization condition $\int(d\Omega_4/V_4)\zeta(\rho)^2 = 1$. Now the mass for $\tilde{A}_\mu$ ($\tilde{A}_z$ is similar) is generated from the third term on the right hand side in (3.8) as

$$\int \frac{d\Omega_4}{V_4}\eta^{\mu\nu}h^{ij}\mathrm{Tr}D_iA_\mu D_jA_\nu = -\frac{1}{2}\epsilon^2\eta^{\mu\nu}\tilde{A}_\mu^a\tilde{A}_\nu^a, \tag{3.11}$$

where a in the upper index denotes SU(2) gauge index. Since the above mass term originates from the commutator, there would be no mass term for the U(1) part even if we consider the U(2) Yang-Mills theory. There are also higher modes on the S^4, which we do not consider in this article. Performing the integration over the S^4 and keeping only terms quadratic in fluctuations, we obtain

$$\mathcal{L} = -\int dz\Big\{\frac{R^3}{4U_z}\eta^{\mu\nu}\eta^{\rho\sigma}\tilde{F}^a_{\mu\rho}\tilde{F}^a_{\nu\sigma} + \frac{9}{8}\frac{U_z^3}{U_{\mathrm{KK}}}\eta^{\mu\nu}\tilde{F}^a_{\mu z}\tilde{F}^a_{\nu z} + \frac{1}{2}\epsilon^2\eta^{\mu\nu}\tilde{A}^a_\mu\tilde{A}^a_\nu + \frac{9}{8}\frac{U_z^4}{R^3U_{\mathrm{KK}}}\epsilon^2\tilde{A}^a_z\tilde{A}^a_z\Big\}. \tag{3.12}$$

Note that if the second term on the left hand side of (3.10) does not exist, then $\zeta =$ constant is the normalizable eigen function with $\epsilon = 0$. If we consider an instanton with large μ, then the second term on the left hand side of (3.10) is small compared to the other two terms for a wide range of the variable ρ, and $\epsilon \sim \mu^{-1}$ as we will see later. Thus for large μ, the third and the fourth terms in (3.12) can be considered as small perturbations added to the model considered by Sakai and Sugimoto.

Next we perform the z-integral. Since we treat the instanton with $\mu^{-1} \ll 1$ as a small perturbation, we may determine mode functions of $\tilde{A}_\mu$ and $\tilde{A}_z$ along the same line as [5].

Although with this choice of mode functions the KK modes are not diagonalized, we can diagonalize them after the z-integral.

Let us first put $\tilde{A}_z = 0$ and determine the mode functions ψ_m for $\tilde{A}_\mu(x,z) = \sum_{m\ge1} \tilde{A}_\mu^{(m)}(x)\psi_m(z)$. The equation for the mode functions which diagonalize the Lagrangian is easily obtained:

$$-\partial_z\big(K\partial_z\psi_m\big) + \frac{4}{9}\epsilon^2 U_{\rm KK}^{-2}\psi_m = \lambda_m U_{\rm KK}^{-2} K^{-1/3}\psi_m \qquad \left(K \equiv \left(\frac{U_z}{U_{\rm KK}}\right)^3\right). \tag{3.13}$$

We choose the normalization condition $\int dz K^{-1/3}\psi_m\psi_n = \delta_{mn}$. As for the mode functions ϕ_m for $\tilde{A}_z(x,z) = \sum_{m\ge0}\tilde{A}_z^{(m)}(x)\phi_m(z)$, we choose

$$\phi_m \equiv \partial_z\psi_m \quad (m\ge1), \qquad \phi_0 \propto \frac{1}{K}, \tag{3.14}$$

as in [5], and in the following we denote the lowest mode as $\tilde{A}_z^{(0)} \equiv \varphi$. Using this mode expansion, we obtain

$$\begin{aligned}
\mathcal{L} =& -\frac{9U_{\rm KK}^2}{8}\left[K_{00}(\partial_\mu\varphi^a)^2 + \frac{4}{9}\epsilon^2 M_{\rm KK}^2 M_{00}(\varphi^a)^2\right] - \frac{R^3}{U_{\rm KK}}\sum_{m\ge1}\left[\frac{1}{4}\big(\tilde{F}_{\mu\nu}^{a,(m)}\big)^2 + \frac{1}{2}\lambda_m M_{\rm KK}^2\big(\tilde{A}_\mu^{a,(m)}\big)^2\right] \\
&-\frac{9U_{\rm KK}^2}{8}\sum_{m,n\ge1}\left[K_{mn}\partial_\mu\tilde{A}_z^{a,(m)}\partial^\mu\tilde{A}_z^{a,(n)} + \frac{4}{9}\epsilon^2 M_{\rm KK}^2 M_{mn}\tilde{A}_z^{a,(m)}\tilde{A}_z^{a,(n)}\right] \\
&+\frac{9}{4}U_{\rm KK}^2\sum_{m,n\ge1}K_{mn}\tilde{A}_\mu^{a,(m)}\partial^\mu\tilde{A}_z^{a,(n)} - \epsilon^2 U_{\rm KK}^2 M_{\rm KK}^2\sum_{m\ge1}M_{0m}\varphi^a\tilde{A}_z^{a,(m)},
\end{aligned} \tag{3.15}$$

with

$$K_{mn} \equiv \int dz K\phi_m\phi_n, \quad M_{mn} \equiv \int dz K^{4/3}\phi_m\phi_n. \tag{3.16}$$

Here the four-dimensional Lorentz indices are contracted by the flat metric. The parameter $M_{\rm KK}$ is defined as $M_{\rm KK} = 3U_{\rm KK}^{1/2}/2R^{3/2}$. With the above choice of ϕ_m, the mixing terms of φ^a with $\tilde{A}_\mu^{a,(m)}$ vanish, i.e., $K_{0m} = 0$. The third line denotes the mixing of the KK modes $\tilde{A}_\mu^{a,(m)}$ with $\tilde{A}_z^{a,(m)}$ and that of φ^a with $\tilde{A}_z^{a,(m)}$. The mixing terms between $\tilde{A}_\mu^{a,(m)}$ and $\tilde{A}_z^{a,(m)}$ can be absorbed by the following field redefinition:

$$\tilde{B}_\mu^{a,(m)} \equiv \tilde{A}_\mu^{a,(m)} - \frac{U_{\rm KK}^2}{\lambda_m}\sum_{n\ge1}K_{mn}\partial_\mu\tilde{A}_z^{a,(n)}, \tag{3.17}$$

and finally we have

$$\begin{aligned}
\mathcal{L} =& -\frac{9U_{\rm KK}^2}{8}\left[K_{00}(\partial_\mu\varphi^a)^2 + \frac{4}{9}\epsilon^2 M_{\rm KK}^2 M_{00}(\varphi^a)^2\right] \\
&-\frac{R^3}{U_{\rm KK}}\sum_{m\ge1}\left[\frac{1}{4}\big(\tilde{F}_{\mu\nu}^{a,(m)}\big)^2 + \frac{1}{2}\lambda_m M_{\rm KK}^2\big(\tilde{B}_\mu^{a,(m)}\big)^2\right] \\
&-\frac{9U_{\rm KK}^2}{8}\sum_{m,n\ge1}\left[K'_{mn}\partial_\mu\tilde{A}_z^{a,(m)}\partial^\mu\tilde{A}_z^{a,(n)} + \frac{4}{9}\epsilon^2 M_{\rm KK}^2 M_{mn}\tilde{A}_z^{a,(m)}\tilde{A}_z^{a,(n)}\right] \\
&-\epsilon^2 U_{\rm KK}^2 M_{\rm KK}^2\sum_{m\ge1}M_{0m}\varphi^a\tilde{A}_z^{a,(m)},
\end{aligned} \tag{3.18}$$

μ^{-1}		0	0.02	0.05	1/13.0	0.1	0.2	1.0
ϵ		0	0.0488	0.120	0.180	0.230	0.423	1.41
$m_{\pi^\pm,\pi^0}$	(140,135)	0	36.4	88.7	132	167	285	624
m_ρ	(776)	(776)	(776)	(776)	(776)	(776)	(776)	(776)
m_{a_1}	(1230)	1189	1188	1186	1183	1179	1162	1046
$m_{\rho'}$	(1465)	1607	1607	1603	1596	1589	1550	1308

Table 1: The result of the numerical calculation of the meson spectrum. The unit is MeV. The results for $\mu^{-1} = 0$ corresponds to the massless QCD [5]. We have calculated matrices K_{mn} and M_{mn} up to $m = n = 5$. We use ρ meson mass $m_\rho = 776$(MeV) as our input parameter.

with

$$K'_{mn} = K_{mn} - \sum_{p\geq 1} K_{mp}\frac{U_{\mathrm{KK}}^2}{\lambda_p}K_{pn}. \tag{3.19}$$

Note that we still have mixing terms in the pion sector which we diagonalize after calculating matrices K_{mn} and M_{mn}. Since $K_{mn} = M_{mn} = 0$ for $(m,n) = (\mathrm{even},\mathrm{odd})$, there are no mixing terms between the odd modes and the even modes. We can see that, after the limit $\epsilon \to 0$ is taken, this Lagrangian becomes the one for the pion and the vector mesons in [5].

Adopting the same charge conjugation and parity assignment as in [5], the lowest mass eigen mode of A_z, i.e. φ with a small mixing with $\tilde{A}_z^{(m)}$, should be identified with the pion, and the modes $\tilde{A}_\mu^{(m)}$ correspond to the vector mesons. The other modes in A_z were ignored in [5] and we follow this rule as we think of our deformation as a perturbation.[9] We expect that the mixing of φ with each mode $\tilde{A}_z^{(m)}$ gets smaller as m becomes larger. We compute the relevant matrices K_{mn} and M_{mn} up to level $m = 5$ and diagonalize the system. In table 1 we summarize our final results of the pion and the vector meson masses. We have used the actual observed ρ-meson mass $m_\rho = 776$ (MeV) as our input parameter. From table 1, we understand that ϵ is of the order μ^{-1} (μ is the instanton size in $\mathbf{R}^4$), and that turning on a small μ^{-1} successfully generates the small mass for the pions. Around $\mu = 13$, the ratio m_ρ/m_π comes close to the ratio of the actual observed masses. On the other hand, the spectrum of the other vector mesons is not much affected compared with that of Sakai and Sugimoto [5].

Although we have succeeded in deriving the correct value of the pion mass, it does not necessarily mean that we have succeeded in adding non-zero quark masses to QCD. It is worth noting that the instanton did not introduce a mass term for the possible Nambu-Goldstone boson associated with the spontaneous symmetry breaking of the axial U(1) part of the $\mathrm{U}(2)_L \times \mathrm{U}(2)_R$ chiral symmetry. This point may suggest that our deformation differs from adding the quark masses. In the next subsection, we consider what kind of perturbation corresponds to the introduction of the instanton from the view point of chiral perturbation. There we will show that considering the property of the A_i^{inst} under the

[9] In this paper we take the gauge fixing condition in which the modes of A_i are eaten by the massive modes of A_μ, on the other hand in [5], the modes of A_z are eaten by the massive modes of A_μ.

chiral symmetry transformation, a four fermi coupling is more natural than the quark mass terms as a possible lowest perturbation to the massless QCD. However one may still wonder whether there is a relation between the length of a stretched string and the quark mass. We will discuss this point further in the appendix D by considering the distance between the D4-branes and connected D8-$\overline{\text{D8}}$ branes in the weak coupling regime, i.e., in the flat spacetime background. Though in this paper we worked for the case of $N_f = 2$, for $N_f > 2$ one can introduce multi-instanton configurations on S^4 to reproduce the observed structure of mass spectrum of π/K.

3.3 D4-branes as a chiral perturbation

Since the pion mass derived in the previous subsection is well below the QCD scale, we should be able to understand the introduction of an instanton in terms of a chiral perturbation. In this subsection we follow the discussion by Sakai and Sugimoto on the derivation of the pion effective action and study the chiral perturbation. As studied in the previous subsection, since the instanton does not affect the pion associated with the axial U(1) breaking, we study a chiral perturbation concerning only the pions associated with the chiral $\mathrm{SU}(N_f)$ breaking.

We define a group element of the chiral symmetry transformation following the paper [5]. In the previous subsection, we have considered fluctuations of the gauge fields which vanish at $z \to \pm\infty$. The gauge transformation with an element $g(x, z, \theta) \in \mathrm{SU}(N_f)$ which approaches a constant in the limit $z \to \pm\infty$ does not change this asymptotic behavior of the gauge fields and is a transformation of residual gauge symmetry. The constant of this transformation is identified as an element of the chiral symmetry transformation $(g_+, g_-) \in \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, $g_\pm \equiv \lim_{z\to\pm\infty} g(x, z, \theta)$. If we introduce the following field $U(x, \theta)$:

$$U(x,\theta) \equiv \mathrm{P}\exp\left\{-\int_{-\infty}^{\infty} dz A_z(x,z,\theta)\right\}, \tag{3.20}$$

it is transformed as $U \to g_+ U g_-^{-1}$ under the chiral transformation. For our later convenience we further introduce

$$\xi_\pm^{-1}(x,\theta) \equiv \mathrm{P}\exp\left\{-\int_0^{\pm\infty} dz' A_z(x,z',\theta)\right\}, \quad A_i^\pm(\theta) \equiv A_i^{\mathrm{inst}}(x, z=\pm\infty, \theta) = A_i^{\mathrm{inst}}(\theta). \tag{3.21}$$

Then we have an expression $U(x,\theta) = \xi_+^{-1}(x,\theta)\xi_-(x,\theta)$ and the transformation rules of $\xi_\pm$ and $A_i^\pm$ under the residual gauge symmetry above are as follows:

$$\xi_+ \to h(x,\theta)\xi_+ g_+^{-1}, \quad \xi_- \to h(x,\theta)\xi_- g_-^{-1}, \quad A_i^+ \to g_+ A_i^+ g_+^{-1}, \quad A_i^- \to g_- A_i^- g_-^{-1}, \tag{3.22}$$

with $h(x,\theta) \equiv g(x, z=0, \theta)$. Note that we can take $h(x,\theta)$ and $g_\pm$ as independent group elements since $g_\pm$ does not fix the gauge transformation parameter at finite z. Thus we can choose the gauge $\xi_- = 1$ (and therefore $\xi_+^{-1} = U$) by using the degree of freedom of $h(x,\theta)$.[10]

[10]Note that none of the group elements $g_\pm$, $h(x,\theta)$ nor $\xi_\pm(x,\theta)$ have nontrivial winding on the S^4.

We move to the $A_z = 0$ gauge using the gauge transformation by the group element $\mathrm{P}\exp(-\int_0^z A_z dz')$. Then the asymptotic behavior of the gauge fields at $z \to \pm\infty$ now becomes

$$A_\mu(x,z,\theta) \to \xi_\pm(x,\theta)\partial_\mu\xi_\pm^{-1}(x,\theta), \qquad A_i(x,z,\theta) \to \xi_\pm(x,\theta)(A_i^\pm(\theta)+\partial_i)\xi_\pm^{-1}(x,\theta), \tag{3.23}$$

respectively. From this behavior and the above mentioned gauge choice, $\xi_- = 1$ and $\xi_+^{-1} = U$, we can expand the fluctuations with respect to the mode functions of z-direction as follows:

$$A_\mu(x,z,\theta) = U^{-1}(x,\theta)\partial_\mu U(x,\theta)\psi_+(z) + \text{higher modes}, \tag{3.24}$$

$$A_i(x,z,\theta) = U^{-1}(x,\theta)(A_i^+(\theta)+\partial_i)U(x,\theta)\widetilde{\psi}_+(z) + A_i^-(\theta)\widetilde{\psi}_-(z) + \text{higher modes}. \tag{3.25}$$

Here, since we treat the instanton as a perturbation, we take $\psi_\pm$ and $\widetilde{\psi}_\pm$ as the zero modes for the case of the absence of the instanton background. The explicit forms of these functions can be read from the action:

$$\psi_\pm(z) = \frac{1}{2}\left(1 \pm \frac{C_{-1}(z)}{C_{-1}(\infty)}\right), \quad \widetilde{\psi}_\pm(z) = \frac{1}{2}\left(1 \pm \frac{C_{-4/3}(z)}{C_{-4/3}(\infty)}\right), \quad C_n(z) = \int_0^z dz K^n. \tag{3.26}$$

In the following, we will neglect the higher modes in the z-space in (3.24) and (3.25), since we are interested only in the pion fields. Using (3.24) and (3.25), the field strengths are computed as

$$F_{\mu\nu} = [U^{-1}\partial_\mu U, U^{-1}\partial_\nu U]\psi_+(\psi_+ - 1), \tag{3.27}$$

$$F_{z\mu} = U^{-1}\partial_\mu U \partial_z \psi_+, \tag{3.28}$$

$$F_{\mu i} = [U^{-1}\partial_\mu U, U^{-1}\mathcal{D}_i U]\psi_+\widetilde{\psi}_+ + U^{-1}D_i^+\left(\partial_\mu U U^{-1}\right)U\widetilde{\psi}_+ - D_i^-\left(U^{-1}\partial_\mu U\right)\psi_+, \tag{3.29}$$

$$F_{zi} = U^{-1}\mathcal{D}_i U \partial_z \tilde{\psi}_+, \tag{3.30}$$

$$F_{ij} = [U^{-1}\mathcal{D}_i U, U^{-1}\mathcal{D}_j U]\widetilde{\psi}_+(\widetilde{\psi}_+ - 1) + U^{-1}F_{ij}^+ U\widetilde{\psi}_+ + F_{ij}^-\widetilde{\psi}_-. \tag{3.31}$$

Here the covariant derivatives and the field strengths are defined by $D_i^\pm * = \partial_i * + [A_i^\pm, *]$, $\mathcal{D}_i U = \partial_i U + A_i^+ U - U A_i^-$ and $F_{ij}^\pm = \partial_i A_j^\pm - \partial_j A_i^\pm + [A_i^\pm, A_j^\pm]$. Let us substitute these expressions into the D8-brane action:

$$\begin{aligned} S_{D8} =& \tilde{T}(2\pi\alpha')^2 \int d^4x dz \frac{d\Omega_4}{V_4}\, 2\mathrm{Tr}\left[\frac{R^3}{4U_z}\eta^{\mu\sigma}\eta^{\nu\tau}F_{\mu\nu}F_{\sigma\tau} + \frac{9U_z^3}{8U_{\mathrm{KK}}}\eta^{\mu\nu}F_{\mu z}F_{\nu z}\right. \\ & \left. + \frac{1}{2}\eta^{\mu\nu}h^{ij}F_{\mu i}F_{\nu j} + \frac{9U_z^4}{8R^3 U_{\mathrm{KK}}}h^{ij}F_{zi}F_{zj} + \frac{U_z}{4R^3}h^{ik}h^{jl}F_{ij}F_{kl}\right]. \end{aligned} \tag{3.32}$$

Among the terms induced by the instanton, the $F_{\mu i}^2$ terms give kinetic terms and interactions, and the F_{zi}^2 and the F_{ij}^2 terms induce the pion mass and interactions. Substituting the expressions of F written in terms of U, we obtain the following action:

$$
\begin{aligned}
S_{D8} = \tilde{T}(2\pi\alpha')^2 \int d^4x dz \frac{d\Omega_4}{V_4}\ 2\mathrm{Tr}\Bigg[& \frac{R^3}{4U_z}[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U]^2 \psi_+^2(\psi_+ - 1)^2 \\
& + \frac{9U_z^3}{8U_{\mathrm{KK}}}(U^{-1}\partial_\mu U)^2(\partial_z\psi_+)^2 + \frac{9U_z^4}{8R^3U_{\mathrm{KK}}}(U^{-1}\mathcal{D}_i U)^2(\partial_z\widetilde{\psi}_+)^2 \\
& + \frac{1}{2}\Big([U^{-1}\partial_\mu U, U^{-1}\mathcal{D}_i U]\psi_+\widetilde{\psi}_+ + U^{-1}D_i^+(\partial_\mu U U^{-1})U\widetilde{\psi}_+ - D_i^-(U^{-1}\partial_\mu U)\psi_+\Big)^2 \\
& + \frac{U_z}{4R^3}\Big([U^{-1}\mathcal{D}_i U, U^{-1}\mathcal{D}_j U]\widetilde{\psi}_+(\widetilde{\psi}_+ - 1) + U^{-1}F_{ij}^+ U\widetilde{\psi}_+ + F_{ij}^-\widetilde{\psi}_-\Big)^2\Bigg]. \qquad (3.33)
\end{aligned}
$$

Here the indices μ, ν and i are contracted by $\eta^{\mu\nu}$ and h^{ij}, respectively.

Let us consider the expansion of the $U(x,\theta)$ field using the mode functions for the case without the instanton. The lowest mode in such expansion is just constant and thus

$$
U(x,\theta) = \exp\big(2i\pi(x)/f_\pi + \text{higher } S^4 \text{ KK modes}\big). \qquad (3.34)
$$

Here $\pi(x) = \pi^a(x)T^a$ is the pion field, which we choose to be Hermitian. If we neglect the higher modes in this expression as $U = U(x) = \exp(2i\pi(x)/f_\pi)$ and substitute it into (3.33), we have the following four-dimensional chiral Lagrangian:

$$
\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}(U^{-1}\partial_\mu U)^2 + C \int \frac{d\Omega_4}{V_4}\mathrm{Tr}\left(U^{-1}A_i^+ U A_i^-\right) + \mathcal{O}(\mu^{-4}). \qquad (3.35)
$$

Here f_π and C are given by

$$
f_\pi^2 \equiv \tilde{T}(2\pi\alpha')^2 \int dz\ \frac{9U_z^3}{U_{\mathrm{KK}}}\big(\partial_z\psi_+\big)^2, \quad C \equiv -\tilde{T}(2\pi\alpha')^2 \int dz \frac{9U_z^4}{2R^3U_{\mathrm{KK}}}(\partial_z\widetilde{\psi}_+)^2. \qquad (3.36)
$$

In (3.35), we show terms of leading order in chiral perturbation theory. In the previous subsection we saw that the magnitude of the effect of the instanton with the size μ can be estimated as $D_i \sim \epsilon \sim \mu^{-1}$ (see (3.11)). Hence we can treat $\mathcal{D}_i \sim D_i^\pm \sim \mu^{-1}$ and $F_{ij}^\pm \sim \mu^{-2}$. In addition, we took into account the fact that usually in chiral perturbation theories one counts the dimension of the momentum ∂_μ in the same manner, $\partial_\mu \sim m_\pi \sim \mu^{-1}M_{\mathrm{KK}}$.

The pion mass term, i.e. the second term in (3.35) does not have the same form as the lowest mass term for pions in the chiral perturbation with non-zero quark masses, $U\chi + \chi^\dagger U^{-1}$ where χ is related to the bare quark masses. The form (3.35) of the chiral Lagrangian suggests that our deformation corresponds to turning on (an infinite number of) external fields $A_i^\pm(\theta)$ in the QCD. From the transformation laws under the chiral transformation in (3.22), we can read out possible lowest terms of the perturbation to the QCD action:

$$
\mathcal{L} = \mathcal{L}_{QCD} + G_{bq}^{ap}\bar{q}_{La}{q_R}^q\bar{q}_{Rp}{q_L}^b + h.c., \qquad (3.37)
$$

where a, b (p, q) are the indices of $\mathrm{SU}(N_f)_L$ $(\mathrm{SU}(N_f)_R)$ and q_L (q_R) is the left-handed (the right-handed) quark field. The tensor G_{bq}^{ap} is related to the two sources A_i^+ and A_i^-. From the symmetry arguments we can only guess possible terms with which the chiral symmetry is explicitly broken and in particular $A_i^{\pm}$ do not couple left- and right- handed quarks in a bilinear manner.

When all the masses for the quarks are the same, the masses of the pions are equal to each other, because $\mathrm{SU}(N_f)_V$ is unbroken. In our case, although both the sources A_i^+ and A_i^- break all the chiral symmetries explicitly because of (3.22), the masses of the pions in (3.35) are the same. This happens due to the fact that the BPST instanton has a symmetric structure among the gauge indices a and the spacetime coordinates i and the contribution of instanton becomes gauge-blind after contracting the Lorentz indices i.

Before ending this subsection we will give some remarks. In the previous subsection, we obtained the lowest S^4 mode, $\zeta(\rho)$, in the presence of instanton. If we use this function in expanding $U(x,\theta)$ as:

$$U(x,\theta) = \exp\Big(2i\pi(x)\zeta(\rho)/\tilde{f}_\pi + \text{ higher } S^4 \text{ KK modes }\Big), \tag{3.38}$$

then the Lagrangian (3.33) does not reduce to the four-dimensional chiral Lagrangian written in terms of U, even if we neglect the higher S^4 KK modes. Inserting (3.38) into (3.33) and expanding it in terms of $\pi(x)$, we have

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \pi^a)^2 - \frac{1}{2}m_\pi^2(\pi^a)^2 + \frac{1}{4\tilde{e}^2\tilde{f}_\pi^2}([\partial_\mu\pi, \partial_\nu\pi]^a)^2 + \cdots . \tag{3.39}$$

Here the parameters are defined as follows:

$$\tilde{f}_\pi^2 \equiv \tilde{T}(2\pi\alpha')^2 \int dz\, \left\{\frac{9U_z^3}{U_{\mathrm{KK}}}\big(\partial_z\psi_+\big)^2 + 4\epsilon^2\big(\tilde{\psi}_+ - \psi_+\big)^2\right\}, \tag{3.40}$$

$$\frac{\tilde{f}_\pi^2}{2\tilde{e}^2} \equiv \tilde{T}(2\pi\alpha')^2 \int dz\, \frac{8R^3}{U_z}\psi_+^2(\psi_+ - 1)^2 \int \frac{d\Omega_4}{V_4}\zeta^4, \tag{3.41}$$

$$m_\pi^2 \equiv \epsilon^2\frac{\tilde{T}(2\pi\alpha')^2}{\tilde{f}_\pi^2} \int dz\, \frac{9U_z^4}{R^3U_{\mathrm{KK}}}\big(\partial_z\tilde{\psi}_+\big)^2. \tag{3.42}$$

In deriving these, we have used the eigenvalue equation and the normalization condition for ζ which we used in the previous subsection. It can be checked that the pion mass (3.42) reproduces the values which are consistent with those in the previous subsection for large μ.

The final point is related with the tachyon field. As discussed in [14], a tachyon field, which is the lowest mode of a string stretching between D8 and $\overline{\mathrm{D8}}$, couples with the quark bilinear, and if it develops a vacuum expectation value (VEV), the quark mass terms will be generated. The tachyon field belongs to the bi-fundamental representation of the chiral symmetry group $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$. Inspired by this, let us add a complex scalar field T on the D8-branes which belongs to a fundamental representation of the group of the gauge symmetry on the D8-branes, and let it develop a VEV $\langle T\rangle$. Following the same procedure the tachyon field will be expanded as

$$T(x,z,\theta) = \xi_+(x,\theta)T_+\Psi_+(z) + \xi_-(x,\theta)T_-\Psi_-(z), \qquad T_+ = T_- = \langle T\rangle, \tag{3.43}$$

with non-normalizable modes $\Psi_\pm$ (where $\Psi_- \equiv 1 - \Psi_+$, $\Psi_+(z = \infty) = 1$ and $\Psi_+(z = -\infty) = 0$). The lowest contribution to the chiral Lagrangian comes from the kinetic term $g^{zz}|\partial_z T|^2$,

$$T_{\rm D8} \int d^9\sigma e^{-\phi}\sqrt{-\det g}g^{zz}{\rm Tr}|U^{-1}T_+ - T_-|^2\big(\partial_z\Psi_+\big)^2 \propto \int d^4x\ {\rm Tr}[U\chi + \chi^\dagger U^{-1}], \quad (3.44)$$

with $\chi = T_- T_+^\dagger$. This is exactly the expected form of the pion mass term induced by the non-zero quark masses in the chiral perturbation theory. The actual tachyon mass profile is complicated and should depend on the spacetime coordinates in a curved geometry (see [9] for a profile of fundamental strings in the D4-brane geometry).

4. Conclusion and discussion

In this paper we have studied deformations of the holographic QCD model considered by Sakai and Sugimoto, in order to introduce explicit chiral symmetry breaking and non-zero pion mass. In the Sakai-Sugimoto model, the chiral symmetry is realized by the D-brane configuration in the weak coupling regime; flavor D8- and $\overline{\rm D8}$-branes are separated from each other. So we have considered deformations of the D8- and $\overline{\rm D8}$-branes to connect them in the flat spacetime.

In section 2, we have considered the configuration in which the D8- and $\overline{\rm D8}$-branes are connected by a throat and color D4-branes are placed in the throat. First we placed the D4-branes at the center of the throat. Because the size of the throat is of the same order as the asymptotic distance between the D8- and $\overline{\rm D8}$-branes, they do not reach the near horizon region of the D4-brane background. Without taking the near horizon limit, we have studied the spectrum of the gauge fields and we still found a massless pion. Of course it is possible that without taking the near horizon limit, this model just fails to capture the strong dynamics of the dual QCD. In order to understand this point further, we took a certain limit of this brane configuration in which the system reduces to the simpler one; We placed the D4-branes close to the D8-branes and magnified around a point on an angular S^4 of the D8-brane worldvolume. Then the system reduces to the flat D4- and the flat D8-branes located parallel to each other. Because the D8-branes can be located arbitrarily close to the D4-branes, we can take the near horizon limit of the D4-brane background keeping the D8-branes in the near horizon region. In this case, we found that the pions successfully acquire non-zero mass. However the minimum value of the pion mass we obtained is too large; about 0.8 times the ρ meson mass. This model has no parameter which can be tuned to take the massless pion limit.

In order to construct a model which reproduces the small pion mass, we considered a different deformation in section 3. We introduced an instanton background on the D8-branes considered by Sakai and Sugimoto. From the point of view of the D-brane configuration, the instanton charge corresponds to a smeared D4-brane charge, and the existence of this charge assures that the D8- and the $\overline{\rm D8}$-branes are connected to each other, even in the flat spacetime. Studying the spectrum of the fluctuations, we found a non-zero pion mass which is tunable using the parameter μ, the size of the instanton, and realized the observed value of the pion mass. We have found also that the vector-meson spectrum is not significantly affected. In section 3.3, we studied a chiral perturbation for QCD which corresponds to introducing the instanton background. Using the mode expansions without taking into account the perturbations induced by the instanton, we have derived the four-dimensional chiral Lagrangian. We also discussed the corresponding perturbations to the QCD Lagrangian. A possible lowest candidate for the perturbation is a four-Fermi coupling, not the quark mass term. Also the fact that the mass term of the Nambu-Goldstone boson for the axial U(1) part has not been generated by the instanton suggests that our deformation differs from adding the non-zero quark mass. Using the lowest S^4 mode in the presence of the instanton, we have obtained the pion masses which are consistent with the values in the subsection 3.2.

A final remark is a discussion on the throat solution in DBI action and the quark mass. If we assume that the DBI action is reliable even for the small D8 throat considered in appendix D, then the possible quark mass estimated from the length of an open string stretching between the D4- and D8-branes is finite. Although the D-brane picture may break down for the thin D8 throat with the radius of the string length, the throat can be thought as the result of partial tachyon condensation [15]. The chiral symmetry is broken due to the condensation and the quark mass terms are expected to be induced since the tachyon couples with quarks bilinearly (see [16] for a recent discussion along this direction). It is deserved to understand more clearly the size of throat in the flat space in terms of weak coupling physics and it is important to understand how to implement this tachyon field into the probe D8-brane theory. We leave this issue as a future problem.

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A. D8 probe in the D4 geometry

In this appendix, we discuss the D8-brane probe solutions in the full non-extremal D4-

brane geometry [17] compactified on an S^1 without taking the near horizon limit. We show that there are a set of solutions which have the same asymptotic behavior. The first solution is the one used by Sakai and Sugimoto and the second solution corresponds to the configuration of D8- and $\overline{\text{D8}}$- branes connected to each other by a throat.

The non-extremal D4-brane geometry is given by

$$ds^2 = \left(\frac{U^3}{R^3+U^3}\right)^{1/2}(dx_4^2 + f(U)d\tau^2) + \left(\frac{U^3}{R^3+U^3}\right)^{-1/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \tag{A.1}$$

$$e^{\phi} = \left(\frac{U^3}{R^3+U^3}\right)^{1/4}, \quad F_4 = \frac{2\pi N_c}{V_4}\epsilon_4, \quad f(U) = 1 - \frac{U_{\text{KK}}^3}{U^3}. \tag{A.2}$$

This geometry is regular if and only if the compact τ direction has a periodicity $\tau \sim \tau + \delta\tau$, $\delta\tau = 4\pi R^{3/2}/3U_{\text{KK}}^{1/2}$. The near horizon geometry (3.1) is obtained by taking the small U/R limit.

We place a probe D8-brane so that its configuration in τ-U plane is given by $\tau = \tau(U)$ or equivalently by $U = \mathrm{U}(\tau)$. With this ansatz, the relevant part of the action is

$$S_{D8} = -T_{D8}\int d^9\sigma e^{-\phi}\sqrt{-\det g} \propto \int d\tau U^4 \sqrt{f(U) + \frac{R^3+U^3}{U^3}\frac{U'^2}{f(U)}}. \tag{A.3}$$

Since the action does not depend on τ explicitly, there is a conserved quantity:

$$\frac{U^4 f(U)}{\sqrt{f(U) + \frac{R^3+U^3}{U^3}\frac{U'^2}{f(U)}}} = U_0^4\sqrt{f(U_0)}, \tag{A.4}$$

where $U_0 = \mathrm{U}(\tau_0) \geq U_{\text{KK}}$ is the point where $U'(\tau_0) = 0$. We can easily solve this equation and obtain

$$\tau(U) - \tau_0 = \int_{U_0}^{U} dU \frac{1}{f(U)}\sqrt{\frac{R^3+U^3}{U^3}\frac{U_0^8 f(U_0)}{U^8 f(U) - U_0^8 f(U_0)}}. \tag{A.5}$$

It is easy to see that as U goes to infinity $\tau(U)$ goes to a constant which gives the asymptotic distance between the D8- and the $\overline{\text{D8}}$-branes, $L = 2\tau(U \to \infty)$ modulo $\delta\tau$. When $U_0 = U_{\text{KK}}$, $L = \delta\tau/2$ and this is the solution considered in [5]. As the difference $U_0 - U_{\text{KK}}$ gets larger, L first decreases, but around a critical value $U_0 - U_{\text{KK}} \sim 7$ (for $R = 10$ and $U_{KK} = 1$), L starts increasing and it continues to increase after passing this value. So, in fact, we found two solutions for fixed $\delta\tau$. For example, for $L = \delta\tau/2$, $U_0 = U_{\text{KK}}$ is an obvious solution found before, and the second solution with $U_0 \sim U_{\text{KK}} + 70$ for $R = 10$ and $U_{KK} = 1$ corresponds to the configuration of the D8- and $\overline{\text{D8}}$- branes connected with each other by a throat. (The other solutions with $L = \delta\tau/2$ have $2\tau(U \to \infty) > \delta\tau$ in the covering space of the S^1, and thus these solutions correspond to the probe D8-branes wrapping more than once around the S^1 direction.) Since the ratio U/R is small in the near horizon region, the second solution exists outside of the near horizon region, and thus it disappears when we take the near horizon limit.

We can now study whether the pions become massive. Although we do not take the near horizon limit, we simply apply the AdS/CFT correspondence for studying if we have

Nambu-Goldstone bosons. We check it by studying the normalizability of the zero mode in the U-direction (or z-direction in the main text) of the gauge fields on the D8-brane since this mode would be identified as the pion. If it is normalizable, the zero mode is the pion. If it is not instead, there is no Nambu-Goldstone boson. In the latter case, the situation may be consistent with the pion being massive because of the explicit chiral symmetry breaking.

The relevant part of the action for the fluctuations of the gauge fields on the D8-brane is

$$\begin{aligned} S =& -T_{D8}(2\pi\alpha')^2 \int d^9\sigma e^{-\phi}\sqrt{-\det g}\frac{1}{4}g^{MN}g^{PQ}F_{MP}F_{NQ} \\ \sim& -\int d^4x dU U^4 \left(\frac{U^3}{R^3+U^3}\right)^{-1/2}\sqrt{\frac{U^8}{U^8 f(U)-U_0^8 f(U_0)}} \\ &\times\left[\left(\frac{U^3}{R^3+U^3}\right)^{-1}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}+2\frac{U^8 f(U)-U_0^8 f(U_0)}{U^8}\eta^{\mu\nu}F_{\mu U}F_{\nu U}\right], \end{aligned} \tag{A.6}$$

where $\eta^{\mu\nu}$ is the four-dimensional Minkowski metric. Then the zero mode in A_U is given by

$$A_U(x,U) = \pi(x)\phi_0(U), \qquad \phi_0(U) \propto \left(\frac{U^3}{R^3+U^3}\right)^{1/2}\frac{1}{\sqrt{U^8 f(U)-U_0^8 f(U_0)}}, \tag{A.7}$$

and it is easy to see that the zero mode is normalizable:

$$\int_{U_0}^{\infty} dU \left(\frac{U^3}{R^3+U^3}\right)^{-1/2}\sqrt{U^8 f(U)-U_0^8 f(U_0)}\phi_0(U)^2 < \infty. \tag{A.8}$$

Thus we always have a massless Nambu-Goldstone boson for arbitrary U_0. Therefore even though the D8- and $\overline{\text{D8}}$-branes are connected by a throat in the flat spacetime, the pion is still massless.

B. D8-brane parallel to D4-brane

In this appendix, we compute the pion mass in the D-brane configuration in which a D8-brane is placed parallel to the color D4-branes. The effective theory is a QCD with massive quarks. Although the masses of the quarks are roughly of the same order as the compactification scale, we may still expect that the pion masses are suppressed compared with the other meson masses.

For our later convenience, we introduce a new coordinate r defined by

$$r = \left(\frac{\sqrt{U^3}+\sqrt{U^3-U_{\rm KK}^3}}{2}\right)^{2/3}, \qquad U = \left(r^{3/2}+\frac{U_{\rm KK}^3}{4r^{3/2}}\right)^{2/3}, \tag{B.1}$$

and the near horizon limit of the D4-brane geometry is

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}(dx_4^2+f(U)d\tau^2)+\left(\frac{R}{U}\right)^{3/2}\frac{U^2}{r^2}dX_5^2, \tag{B.2}$$

$$dX_5^2 = dy^2+y^2 d\Omega_3^2+dw^2, \qquad r^2 = y^2+w^2. \tag{B.3}$$

We introduce the D8-brane in such a way that it is parallel to the D4-branes and thus it is localized in the w-direction. The position in the w-direction becomes a function of the y coordinate, $w = w(y)$. Then the induced metric and the equation of motion for $w(y)$ computed from Dirac-Born-Infeld action for the D8-brane probe are given by

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} (dx_4^2 + f(U)d\tau^2) + \left(\frac{R}{U}\right)^{3/2} \frac{U^2}{r^2}(dy^2 + y^2 d\Omega_3^2 + w'(y)^2 dy^2), \tag{B.4}$$

$$w''(y) = (1 + w'(y)^2) \left(-\frac{3}{y} w'(y) + \frac{3U_{\rm KK}^3}{4r^5} \left(w(y) - y w'(y) \right) \left(\frac{1}{1 - \frac{U_{\rm KK}^3}{4r^3}} - \frac{5}{3} \frac{1}{1 + \frac{U_{\rm KK}^3}{4r^3}} \right) \right). \tag{B.5}$$

The solution $w(y)$ of this equation satisfies the asymptotic behavior

$$w(y) \sim m + \frac{\nu(m)}{y^2}, \tag{B.6}$$

where $\nu(m)$ is determined so that the solution $w(y)$ is regular everywhere, and m is a parameter which describes the asymptotic distance between the D4- and D8-branes. Since the quark masses receive corrections through integrating out massive Kaluza-Klein modes along the τ direction, there is no identification that m is the mass of the quark (at the compactification scale) and $\nu(m)$ is proportional to the value of the chiral condensate.

The fluctuations in A_μ on the D8-branes are identified as vector mesons in QCD and so the lowest mode is identified as the ρ meson. We can safely take the $A_y = 0$ gauge since there is no scalar zero mode among the fluctuations of A_μ. Then the lowest Kaluza-Klein mode in the fluctuation A_τ can be identified as the massive pion. The action for fluctuations on the D8-brane at the quadratic level is given by

$$\begin{aligned} S =& - T_{D8}(2\pi\alpha')^2 \int d^9\sigma e^{-\phi} \sqrt{-\det g} \frac{1}{4} g^{MN} g^{PQ} F_{MP} F_{NQ} \\ \propto& - \int dy \left(1 - \frac{U_{\rm KK}^3}{4r^3}\right) \left(1 + \frac{U_{\rm KK}^3}{4r^3}\right)^{5/3} y^3 \sqrt{1 + w'^2} \left\{ \left(\frac{R}{U}\right)^3 \eta^{\mu\rho} \eta^{\nu\tau} F_{\mu\nu} F_{\rho\tau} \right. \\ & \left. + \frac{2r^2}{U^2(1 + w'^2)} \eta^{\mu\nu} F_{\mu y} F_{\nu y} + \frac{2R^3}{U^3 f(U)} \eta^{\mu\nu} F_{\mu\tau} F_{\nu\tau} + \frac{2r^2}{f(U) U^2 (1 + w'^2)} F_{y\tau}^2 \right\}, \end{aligned} \tag{B.7}$$

where we have considered only the S^3-independent modes and omitted the S^3 components of the gauge fields. Performing the fluctuation analysis we have numerically obtained the result that the mass ratio m_π/m_ρ is about 0.8. In the numerical calculation, we have chosen the asymptotic distance between the D4- and the D8-branes such that the D8-brane is located close to the D4-branes, but does not reach the point $U = U_{\rm KK}$. The reason for this choice is the following; If the D8-brane is far away from the D4-branes the bare quark mass is large, on the other hand if the D8-brane intersects with the D4-branes, a large correction to the quark mass is expected. In fact the ratio m_π/m_ρ becomes the smallest value for this choice. Therefore the pion mass is much larger than the actual value in the real QCD.

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C. Mixing between lowest S^4 KK modes of A_i with those of A_ν and A_z

In this appendix, we discuss the mixing terms between the lowest S^4 KK modes of the fluctuations A_ν and A_z with those of A_i in the case with the instanton background.

The following mixing terms arise from the terms $F_{\nu i}^2$ and F_{zi}^2 in the Lagrangian:

$$F_{\nu i}^2: \quad \int d^4X L(\rho) \mathrm{Tr} D_\alpha A_\nu \partial_\nu A_\alpha, \qquad F_{zi}^2: \quad \int d^4X L(\rho) \mathrm{Tr} D_\alpha A_z \partial_z A_\alpha, \tag{C.1}$$

where $D_\alpha A = \partial_\alpha A + [A_\alpha^{\rm inst}, A]$ and $L(\rho) = 4(\rho^2+1)^{-2}$. The index $\alpha = 1, \ldots, 4$ is that for the coordinates X^α of the $\mathbf{R}^4$ introduced in subsection 3.1 and $\rho = |X|$ as before. The lowest modes of the fluctuations A_α are given by the moduli $G_\alpha^{{\rm tr}\beta}(\mathrm{X})$ and $G_\alpha^{\rm size}(\mathrm{X})$ of the instanton which are the translation of the instanton and the change of the instanton size respectively.[11] Using these, the lowest modes of A_α are given by

$$A_\alpha(x,z,X) = f_\beta(x,z) G_\alpha^{{\rm tr}\beta}(X) + f_{\rm size}(x,z) G_\alpha^{\rm size}(X). \tag{C.2}$$

The explicit forms of the translational moduli $G_\alpha^{{\rm tr}\beta}$ and the size modulus $G_\alpha^{\rm size}$ are as follows:

$$G_\alpha^{{\rm tr}\beta}(X) \equiv \frac{\partial}{\partial X^\beta} A_\alpha^{\rm inst}(X) = i\frac{-\epsilon_{\alpha\beta c}\sigma_c + \delta_{\alpha 4}\sigma_\beta - \delta_{\beta 4}\sigma_\alpha}{\mu^2+\rho^2} - \frac{2X^\beta}{\mu^2+\rho^2} A_\alpha^{\rm inst}(X), \tag{C.3}$$

$$G_\alpha^{\rm size}(X) \equiv \frac{\partial}{\partial \mu} A_\alpha^{\rm inst}(X) = -\frac{2\mu}{\mu^2+\rho^2} A_\alpha^{\rm inst}(X). \tag{C.4}$$

Here the parameter μ controls the size of the instanton. The totally antisymmetric tensor $\epsilon_{\alpha\beta\gamma}$ is defined by $\epsilon_{123} = 1$ and $\epsilon_{4\alpha\beta} = 0$ and $\sigma_4 = 0$. Note that the translational moduli $G_\alpha^{{\rm tr}\beta}(X)$ are even under the four-dimensional parity transformation $\{X^\gamma\} \to \{-X^\gamma\}$. Then the mixing term in $F_{\nu i}^2$ which includes $\partial_\nu f_\beta(x,z)$ becomes

$$\int d^4X L(\rho) \mathrm{Tr}\left[G_\alpha^{{\rm tr}\beta}(X) \left\{ \tilde{A}_\nu(x,z) \frac{X^\alpha}{\rho} \partial_\rho \zeta(\rho) + \left[A_\alpha^{\rm inst}, \tilde{A}_\nu(x,z)\zeta(\rho)\right] \right\}\right], \tag{C.5}$$

where the lowest fluctuation modes of A_ν are given by $A_\nu(x,z,X) = \tilde{A}_\nu(x,z)\zeta(\rho)$ as in subsection 3.2. Since the inside of the square bracket is odd under the parity transformation $\{X^\gamma\} \to \{-X^\gamma\}$, this mixing term vanishes after the integration over the $\mathbf{R}^4$. On the other hand, the mixing term in $F_{\nu i}^2$ which includes $\partial_\nu f_{\rm size}(x,z)$ becomes

$$\int d^4X L(\rho) \mathrm{Tr}\left[G_\alpha^{\rm size}(X) \left\{ \tilde{A}_\nu(x,z) \frac{X^\alpha}{\rho} \partial_\rho \zeta(\rho) + \left[A_\alpha^{\rm inst}, \tilde{A}_\nu(x,z)\zeta(\rho)\right] \right\}\right], \tag{C.6}$$

and it vanishes since the size modulus $G_\alpha^{\rm size}(X)$ is proportional to the instanton $A_\alpha^{\rm inst}$. Similarly we can also show that the mixing terms from F_{zi} vanish. Thus there is no mixing between the lowest S^4 KK modes of the fluctuations A_ν and A_z with those of A_i.

[11] The other moduli corresponding to gauge directions can be absorbed by appropriate gauge transformations and field redefinitions.

D. Small throat solution in Dirac-Born-Infeld action

Our original motivation for studying the instanton background was, as is mentioned in the introduction of section 3, that the instanton charge would make the D8-brane and the anti-D8-brane to connect even in the flat spacetime background, giving the explicit breaking of the chiral symmetry. In this appendix, we employ a Dirac-Born-Infeld action as an effective action of the D8-branes in the flat spacetime background, and estimate the throat radius of a classical solution of it when the D4-brane charge (the instanton on the S^4) is introduced. This would correspond to computing the quark mass, from the length of the string stretching between the D8 throat surface and the N_c D4-branes. One should note that the DBI approximation employed here is not a good low energy approximation, especially around the thin throat in the flat spacetime background. It is possible that the quark mass may not be generated from the throat at all, as is in fact indicated in our results of the chiral perturbation. Nevertheless, it is instructive to study, in the weak coupling regime, what would happen to the D8-branes when the D4-brane charge is introduced, by using a DBI action which is at least the best available starting-point in our knowledge.

Our situation is similar to the one in [13] where a set of N D1-branes ending on n D5-branes is obtained as a deformation of the surface of the D5-branes. We take T-dualities three times along the transverse directions, and obtain our brane configuration of the N D4-branes ending on the n D8-branes. In this paper we took $N = 1$ and $n = 2$ (one instanton on the S^4 in the SU(2) Yang-Mills).

The difference from [13] is only that [13] treats infinitely long D4-branes while in our case the D4-brane is ending on the $\overline{\text{D8}}$-branes. In [6], two throat solutions connecting the D8- and the $\overline{\text{D8}}$-branes are obtained, one has the throat whose radius is almost equal to the D8-$\overline{\text{D8}}$ distance, while the other has a small radius of the size of string length. What we are interested in is the latter one, which exists only when one introduces the D4-brane charge on the D8-branes (in [6] fundamental string charge is introduced, instead). Let us construct this latter type of the solution in our situation.

Following (43) and (45) of [13], the equation of motion for the scalar field τ of the D8-branes in the flat background spacetime is written as

$$\frac{d\tau/dU}{\sqrt{1+(d\tau/dU)^2}} = \frac{(2\pi\alpha')^2/\tilde{B}}{nU^4 + (3N/2)(2\pi\alpha')^2}. \tag{D.1}$$

Here we assumed that the N instantons on the D8-brane surface are homogeneous on the S^4 (corresponding to $\mu = 1$ in this paper). Note that we use the same U as the radial coordinate of the D8-branes, *in the flat spacetime*. $\tilde{B}$ is an integration constant. The throat radius U_{throat} can be defined where the derivative $d\tau/dU$ diverges, that means that the above equation is equal to the unity.

The distance between the D8-branes and the $\overline{\text{D8}}$-branes is in our case half the circumference of the S^1, $\delta\tau/2$. Integrating the above equation, we obtain

$$\frac{\delta\tau}{2} = 2\int_{U_{\text{throat}}}^{\infty} dU \ \left(\tilde{B}^2\left(nU^4/(2\pi\alpha')^2 + 3N/2\right)^2 - 1\right)^{-1/2}. \tag{D.2}$$

This integral equation determines the relation between $U_{\rm throat}$ and the integration constant $\widetilde{B}$. We evaluate this in the low energy limit $\alpha' \to 0$ while $\delta\tau$ fixed. This is in fact the limit similar to the one taken in [6], and we apply their choice of the integration constant so that we have the small radius of the throat:

$$\tilde{B} = \frac{2}{3N}(1-\epsilon), \quad \epsilon \to +0(\propto \alpha'^2). \tag{D.3}$$

Then, up to overall numerical constants, we obtain

$$U_{\rm throat} \sim (\alpha')^{1/2}\epsilon^{1/4}, \quad \delta\tau \sim (\alpha')^{1/2}\epsilon^{-1/4}. \tag{D.4}$$

Therefore for a fixed $\delta\tau$ and in the low energy limit $\alpha' \to 0$, the throat radius is estimated as

$$U_{\rm throat} \sim \alpha'/\delta\tau. \tag{D.5}$$

This provides the naive evaluation of the energy of the lowest excitation of the string stretched between the throat surface and the throat center (the N_c D4-branes), which is expected to be the quark mass, as

$$m_{\rm quark} \sim U_{\rm throat}\alpha'^{-1} \sim 1/\delta\tau. \tag{D.6}$$

Here the string tension is proportional to α'^{-1}. We find that the quark mass is finite in the low energy limit $\alpha' \to 0$.

In order to see whether the Gell-Mann–Oakes–Renner (GOR) relation $m_\pi^2 = Bm_q$ (where B is related to the quark condensate) is satisfied, let us evaluate the μ dependence of the quark mass from this thin throat picture quantitatively. When we deform the instanton distribution away from the homogeneous one, it is expected that the spherical cross-section S^4 is deformed in the analysis in the flat spacetime background.[12] Thus the effective length of the stretched string is expected to be shortened by this deformation, and the quark mass may become lighter. To see this concretely, we consider the case with two instantons on the S^4; one on the north pole and the other on the south pole, in order to maintain the parity symmetry $\theta^1 \leftrightarrow \pi - \theta^1$ on the deformed shape of the S^4, for simplicity. The size of each instanton is given by $1/\mu$. For large μ, the instanton density at the poles is $F^2 \sim \mu^4$ while that around the equator is $F^2 \sim 1/\mu^4$. To maintain the force balance on the S^4, the D8-brane shape should be deformed in such a way that the out-bound force generated by this instanton density is canceled by the in-bound force given by the D8-brane energy density per unit angular volume on the deformed S^4. Due to the parity symmetry, we deform the S^4 to an ellipsoid whose major axis is generated by a multiplication by a factor α while its minor is by a factor β. Then, to keep the tension balance, one would need to require $\alpha \sim \mu$ and $\beta \sim 1/\mu$, because of the instanton energy density. The minor axis has

[12] In the strong coupling regime, the shape of the S^4 is deformed, but this is an effect of higher order in α' and thus we have neglected it. The shape of the D8-brane is non-singular even in the $\alpha' \to 0$ limit. On the other hand, in the weak coupling regime (the flat spacetime), the shape becomes singular in the $\alpha' \to 0$ limit (the angular S^4 shrinks), thus the deformation should be taken into account from the first place.

the length proportional to $1/\mu$, so the quark mass is expected to scale as $m_q \sim 1/(\mu\delta\tau)$. We find that this scaling is different from what we obtained in this paper, $m_\pi \propto \epsilon \propto 1/\mu$, if we apply the GOR relation.

Our estimation of the deformation of the S^4 would have been too naive. Furthermore, we have assumed that the quark mass can be given just by the shortest radius of the throat, which would be incorrect, since the D8-brane surface is highly curved.

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